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Performance of Random Space-Time Precoded Integer Forcing over Compound MIMO Channels

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***Abstract*—The performance of precoded integer forcing equalization for communication over the compound multiple-input multiple-output channel is investigated. It is known that an integer-forcing receiver applied to streams precoded using the encoding matrix of a perfect linear dispersion space-time code achieves capacity up to a constant gap. Hence, it attains an arbitrarily large fraction of capacity for sufficiently large rates. It was recently shown that allowing for a small outage probability, the integer-forcing receiver achieves a large fraction of capacity when random unitary precoding is used over the spatial dimension only. This work demonstrates that further improvement can be achieved by extending the random unitary precoding to the temporal domain. Further, it is shown that at moderate transmission rates, such precoding outperforms perfect linear dispersion space-time precoding.**

I. Introduction

This paper addresses communication over a compound multiple-input multiple output (MIMO) channel, where the transmitter only knows the number of transmit antennas and the mutual information. More specifically, the goal of this work is to assess the performance of practical communication schemes for this scenario, and in particular the performance of precoded integer forcing equalization.

Communication over the compound MIMO channel using an architecture employing space-time linear processing at the transmitter side and integer-forcing (IF) equalization at the receiver side was proposed in [1]. It was shown that such an architecture *universally* achieves capacity up to a constant gap, provided that the precoding matrix corresponds to a linear perfect space-time code ([2], [3]).

Recently, in [4], the outage probability of IF when using random unitary precoding applied over the spatial dimension only was considered and an explicit universal bound on the outage probability for a given target rate was derived.

The present work extends the framework of [4] by considering space-time random unitary precoding, and empirically quantifies its performance.

II. Problem Formulation and Preliminaries

A. Channel Model

The (complex) MIMO channel is described by[1]

$$\boldsymbol{y}_c = \mathbf{H}_c \boldsymbol{x}_c + \boldsymbol{z}_c, \tag{1}$$

where $\mathbf{x}_c \in \mathbb{C}^{N_t}$ is the channel input vector, $\mathbf{y}_c \in \mathbb{C}^{N_r}$ is the channel output vector, $\mathbf{H}_c$ is an $N_r \times N_t$ complex channel matrix, and $\boldsymbol{z}_c$ is an additive noise vector of i.i.d. unit-variance circularly symmetric complex Gaussian random variables. We assume that the channel is fixed throughout the whole transmission period. The input vector $\boldsymbol{x}_c$ is subject to the power constraint[2]

$$\mathbb{E}(\boldsymbol{x}_c^H \boldsymbol{x}_c) \leq N_t \cdot \mathsf{SNR},$$

where without loss of generality we fix $\mathsf{SNR} = 1$.

Consider the mutual information achievable with a Gaussian isotropic or "white input" ([5])

$$C_{\mathrm{WI}} = \log\det\left(\mathbf{I} + \mathbf{H}_c \mathbf{H}_c^H\right). \tag{2}$$

We may define the set

$$\mathbb{H}(C_{\mathrm{WI}}) = \left\{\mathbf{H}_c \in \mathbb{C}^{N_r \times N_t} : \log\det\left(\mathbf{I} + \mathbf{H}_c \mathbf{H}_c^H\right) = C_{\mathrm{WI}}\right\}, \tag{3}$$

of all channel matrices with the same WI mutual information C_{WI}. The corresponding compound channel model is defined by (1) with the channel matrix $\mathbf{H}_c$ arbitrarily chosen from the set $\mathbb{H}(C_{\mathrm{WI}})$. The matrix $\mathbf{H}_c$ is known to the receiver, but not to the transmitter. Clearly, the capacity of this compound channel is C_{WI}, and is achieved with a white Gaussian input.

Denote the achievable rate for a given channel matrix $\mathbf{H}_c$ with an IF receiver as $R_{\mathrm{IF}}(\mathbf{H}_c)$. An explicit expression for this rate is given in Section II-B. Since applying a precoding matrix $\mathbf{P}_c$ results in an effective channel $\mathbf{H}_c \cdot \mathbf{P}_c$, it follows that the achievable rate of IF for this channel is $R_{\mathrm{IF}}(\mathbf{H}_c \cdot \mathbf{P}_c)$. Therefore, following [4], we define the worst-case (WC) outage probability of IF as

$$P_{\mathrm{out}}^{\mathrm{WC}}(C_{\mathrm{WI}}, R) = \sup_{\mathbf{H}_c \in \mathbb{H}(C_{\mathrm{WI}})} \Pr\left(R_{\mathrm{IF}}(\mathbf{H}_c \cdot \mathbf{P}) < R\right), \tag{4}$$

where the probability is with respect to the ensemble of precoding matrices.

B. Integer-Forcing Equalization: Achievable Rates

The architecture of an IF receiver (with successive interference cancellation) is depicted in Figure 1. For our purposes, it will suffice to only state the achievable rates of the scheme.

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[1]We denote all complex variables with c to distinguish them from their real-valued representation.

[2]We denote by $[\cdot]^T$, the transpose of a vector/matrix and by $[\cdot]^H$, the Hermitian transpose.

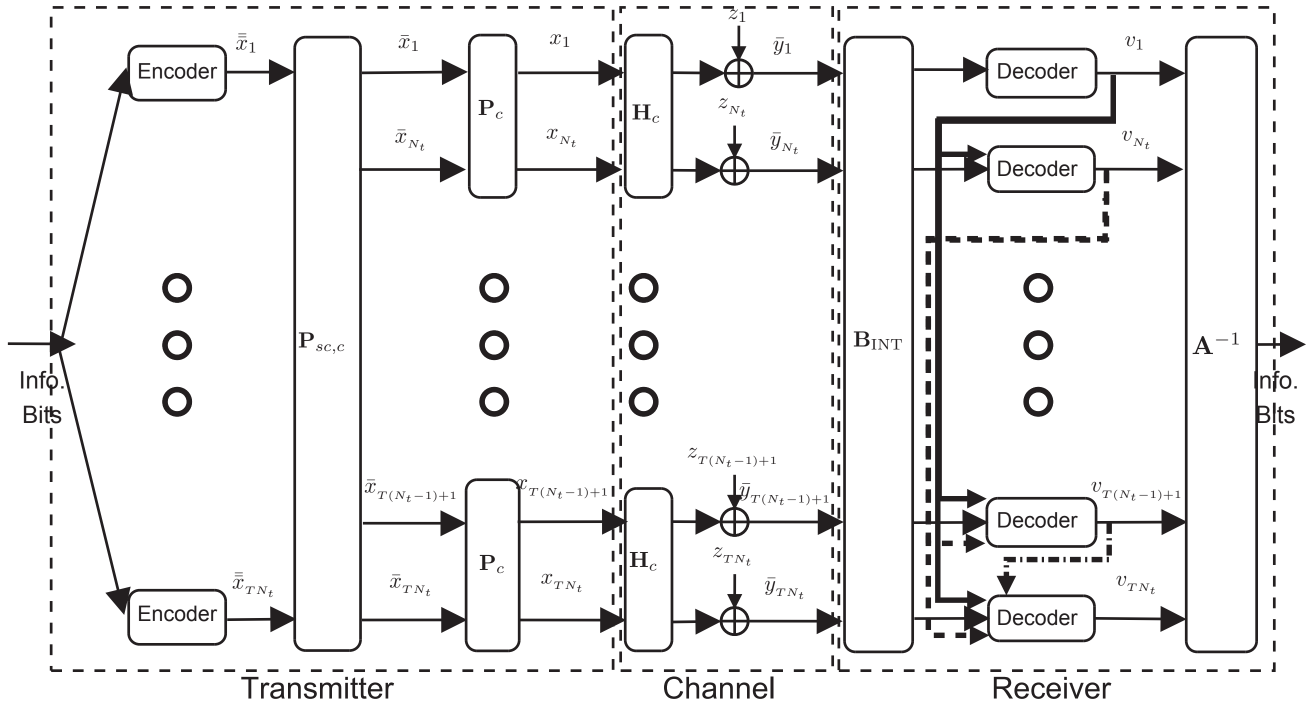

Fig. 1. Block diagram of precoded integer forcing with successive interference cancellation.

The reader is referred to [6] and [7] for the derivation, details and proofs.

We follow the derivation of [6] and describe integer forcing over the reals. Channel model (1) can be expressed via its real-valued representation as

$$\underbrace{\begin{bmatrix}\mathrm{Re}(\mathbf{y}_c)\\ \mathrm{Im}(\mathbf{y}_c)\end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix}\mathrm{Re}(\mathbf{H}_c) & -\mathrm{Im}(\mathbf{H}_c)\\ \mathrm{Im}(\mathbf{H}_c) & \mathrm{Re}(\mathbf{H}_c)\end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix}\mathrm{Re}(\mathbf{x}_c)\\ \mathrm{Im}(\mathbf{x}_c)\end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix}\mathrm{Re}(\mathbf{z}_c)\\ \mathrm{Im}(\mathbf{z}_c)\end{bmatrix}}_{\mathbf{z}}.$$

This real-valued representation is used in the sequel to derive performance bounds for the complex channel $\mathbf{H}_c$. Note that the dimensions of $\mathbf{H}$ are $2N_r \times 2N_t$.

Any real MIMO channel can be described via its singular-value decomposition (SVD) $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$. The following eigenvalue decomposition can thus be easily derived

$$(\mathbf{I} + \mathbf{H}^T\mathbf{H})^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^T, \tag{5}$$

where $\mathbf{D} = \mathbf{I} + \mathbf{\Sigma}^2$. It is shown in [6] that for a given integer coefficient vector $\mathbf{a}_m$, the maximal possible rate for decoding the associated linear combination of messages is

$$R_{\mathrm{IF}}^{\mathbf{a}_m}(\mathbf{H}) = R_{\mathrm{IF}}^{\mathbf{a}_m}(\mathbf{D},\mathbf{V}) = -\frac{1}{2}\log\left(\|\mathbf{D}^{-1}\mathbf{V}^T\mathbf{a}_m\|^2\right). \tag{6}$$

By Theorem 3 in [6], transmission with IF equalization can achieve any rate satisfying $R < R_{\mathrm{IF}}(\mathbf{H})$ where

$$R_{\mathrm{IF}}(\mathbf{H}) = -N_t \log\left(\min_{\substack{\mathbf{A}\in\mathbb{Z}^{2N_t\times 2N_t}\\ \det\mathbf{A}\neq 0}} \max_m \boldsymbol{a}_m^T(\mathbf{I}+\mathbf{H}^T\mathbf{H})^{-1}\boldsymbol{a}_m\right),$$

and where $\mathbf{A} = [\boldsymbol{a}_1 \cdots \boldsymbol{a}_{N_t}]^T$.

We also consider a generalized version of IF that incorporates successive interference cancellation (SIC), which we will refer to as IF-SIC. We next describe the rates achievable with this receiver, where the transmission side remains unchanged.

For a given choice of integer matrix $\mathbf{A}$, let $\mathbf{L}$ be defined by the following Cholesky decomposition

$$\mathbf{A}\left(\mathbf{I}+\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T = \mathbf{A}\mathbf{V}\mathbf{D}^{-1}\mathbf{V}^T\mathbf{A}^T \tag{7}$$

$$= \mathbf{L}\mathbf{L}^T. \tag{8}$$

Denoting by $\ell_{m,m}$ the diagonal entries of $\mathbf{L}$, IF-SIC can achieve [7] any rate satisfying $R < R_{\mathrm{IF-SIC}}(\mathbf{H})$ where

$$R_{\mathrm{IF-SIC}}(\mathbf{H}) = 2N_t\frac{1}{2}\max_{\mathbf{A}} \min_{m=1,\ldots,2N_t} \log\left(\frac{1}{\ell_{m,m}^2}\right) \tag{9}$$

$$= R_{\mathrm{IF-SIC}}(\mathbf{D},\mathbf{V}), \tag{10}$$

and the maximization is over all full-rank integer $2N_t \times 2N_t$ matrices.

C. Space-Only Precoded Integer-Forcing

To understand the role of random precoding on outage probability, we begin with some simple observations.

For conventional linear equalization, bad channels correspond to ill-conditioned matrices. As a concrete example, consider transmission over the 2×2 channel

$$\mathbf{H}_{\mathrm{c,WORST}} = \sqrt{2^8-1}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}. \tag{11}$$

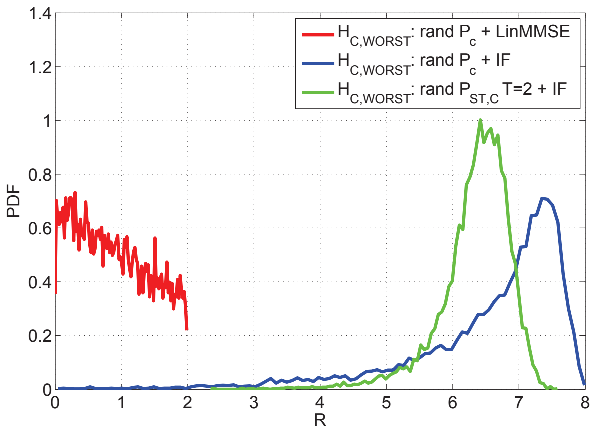

Fig. 2. Approximate probability density function (based on Monte Carlo simulation) of achievable rates of the linear MMSE and IF receivers over the channel (11), applying a random precoding matrix drawn from the CUE.

This channel belongs to the set of channels $\mathbb{H}(C_{\text{WI}} = 8)$ defined in (3). Here, only one transmit and one receive antenna are active so the system reduces to a single-input single-output channel. Thus, the data stream sent from the nonactive antenna is lost. Clearly, in this example, no receiver (including maximum likelihood) will be able to recover the lost data stream. It follows that the achievable rate of both linear and IF equalization is also zero.

Consider now the channel $\mathbf{H}_{c,\text{WORST}} \cdot \mathbf{P}_c$ where $\mathbf{P}_c$ is a unitary matrix. As the singular values remain unchanged, it is clear that the channel remains ill-conditioned and a linear receiver will still suffer from poor performance. On the other hand, IF receivers perform well even over ill-conditioned MIMO channels, and in fact, the performance of the IF receiver for the channel (11) greatly improves for "most" precoding matrices as illustrated below.

Figure 2 compares the achievable rates of the linear MMSE and IF receivers over the channel (11), when applying a random precoding matrix drawn from the circular unitary ensemble (CUE), as will be assumed to be the case throughout this paper. As can be seen, the achievable rate of IF drastically increases for most precoding matrices, achieving a large fraction of C_{WI} with high probability.

III. Space-Time Precoding

A. Background

Combining space-time precoding and integer forcing was suggested in [8], as we next briefly recall.

A block of T channel uses is processed jointly so that the $N_r \times N_t$ physical MIMO channel (1) is transformed into a "time-extended" $N_r T \times N_t T$ MIMO channel. A unitary precoding matrix $\mathbf{P}_{st,c} \in \mathbb{C}^{N_t T \times N_t T}$ that can be either deterministic or random is then applied to the time-extended channel. At the receiver, IF equalization is employed.

We note that, in our framework, two levels of precoding are applied. The first is applied to the physical channel as described in the previous section, and second is applied to the time-extended channel. Hence, the equivalent channel takes the form

$$\mathcal{H}_c^P = \mathbf{I}_{T\times T} \otimes \mathbf{H}_c\mathbf{P}_c = \begin{bmatrix} \mathbf{H}_c\mathbf{P}_c & 0 & \cdots & 0 \\ 0 & \mathbf{H}_c\mathbf{P}_c & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{H}_c\mathbf{P}_c \end{bmatrix}. \quad (12)$$

Let $\bar{\boldsymbol{x}}_c \in \mathbb{C}^{N_t T \times 1}$ be the input vector to the time-extended channel. It follows that the output of the time-extended channel is given by

$$\bar{\boldsymbol{y}}_c^P = \mathcal{H}_c^P \mathbf{P}_{st,c} \bar{\boldsymbol{x}}_c + \bar{\boldsymbol{z}}_c, \quad (13)$$

where $\bar{\boldsymbol{z}}_c$ is i.i.d. unit-variance circularly symmetric complex Gaussian noise. As we assume that both precoding matrices are unitary, the WI mutual information of this channel (normalized per channel use) remains unchanged, i.e.,

$$\frac{1}{T}\log\det\left(\mathbf{I} + (\mathcal{H}_c^P \mathbf{P}_{st,c})(\mathcal{H}_c^P \mathbf{P}_{st,c})^H\right) = C_{\text{WI}}(\mathbf{H}). \quad (14)$$

When using a given space-time precoding ensemble, the WC scheme outage is defined as

$$P_{\text{out}}^{\text{WC}}(C_{\text{WI}}, R) = \sup_{\mathbf{H}_c \in \mathbb{H}(C_{\text{WI}})} \Pr\left(\frac{1}{T} R_{\text{IF}}(\mathcal{H}_c^P \cdot \mathbf{P}_{st,c}) < R\right), \quad (15)$$

Similarly, the corresponding ε-outage capacity of the integer forcing receiver $R_{\text{IF}}(\mathbf{P}_{st,c}; \varepsilon)$ is defined as the rate for which

$$P_{\text{out}}^{\text{WC}}\left(C_{\text{WI}}, R_{\text{IF}}(\mathbf{P}_{st,c}; \varepsilon)\right) = \varepsilon. \quad (16)$$

Finally, the transmission efficiency of a precoding scheme, at a given outage probability ε and WI mutual information C_{WI}, is defined as

$$\eta_\varepsilon(C_{\text{WI}}, \mathbf{P}_{st,c}) = \frac{R_{\text{IF}}(\mathbf{P}_{st,c}; \varepsilon)}{C_{\text{WI}}}. \quad (17)$$

B. Upper Bound - Maximum-Likelihood Decoding

It is natural to compare the performance attained by an IF receiver with that of an optimal maximum likelihood (ML) decoder for the same precoding scheme but where each stream is coded using an independent Gaussian codebook.

Let $\mathbf{H}_S$ denote the submatrix of $\mathbf{H}_c\mathbf{P}_c$ formed by taking the columns with indices in $S \subseteq \{1, 2, ..., N_t\}$. For a joint ML decoder, the following rate is achievable:

$$R_{\text{JOINT}} = \min_{S \subseteq \{1,2,\ldots,N_t\}} \frac{N_t}{|S|} \log\det\left(\mathbf{I}_S + \mathbf{H}_S \mathbf{H}_S^H\right). \quad (18)$$

Note that since $\mathbf{H}_S$ depends on the random precoding matrix $\mathbf{P}_c$, R_{JOINT} is a random variable.

In a similar manner, when using precoding over a time-

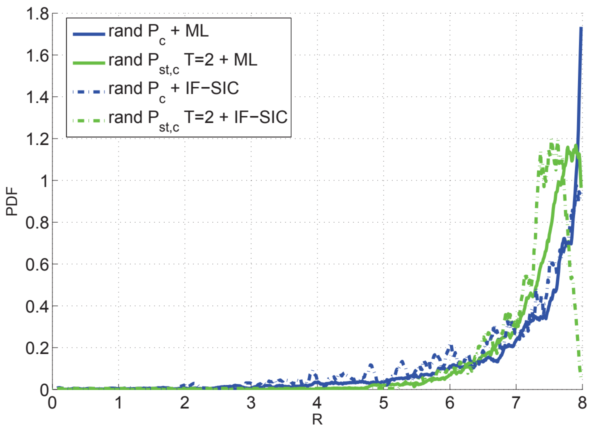

Fig. 3. Approximate worst-case probability density functions (based on Monte carlo simulation) of rates achievable with a joint ML receiver and with an IF-SIC receiver, when applying random CUE precoding over one and two channel uses. The WI mutual information here is $C_{\mathrm{WI}} = 8$.

extended channel, the following rate is achievable:

$$R_{\mathrm{JOINT,ST}} = \frac{1}{T} \min_{S \subseteq \{1,2,\ldots,N_t T\}} \frac{N_t T}{|S|} \log\det\left(\mathbf{I}_S + \mathbf{H}_S \mathbf{H}_S^H\right). \quad (19)$$

where $\mathbf{H}_S$ now corresponds to a submatrix of $\mathcal{H}_c^P \mathbf{P}_{st,c}$.

Figure 3 shows empirical (Monte Carlo) results of the probability density function (PDF) of the worst-case (over all channels in the compound class) rate achieved with random CUE precoding and ML decoding. For each vector of possible singular values, a large number of random unitary matrices was drawn and the cumulative distribution function (CDF) of the corresponding rate was calculated. Then, the worst case (minimum) of these CDFs was found, from which the depicted PDF is derived.

C. Candidate Precoding Schemes

We consider several candidate schemes for the precoding matrix $\mathbf{P}_{st,c}$, some of which are random ensembles, others deterministic.

Orthogonal space-time block code. For low transmission rates it is natural to employ a space-time block code (OSTBC) [9]. As this form of modulation effectively transforms the MIMO channel to a scalar one, integer forcing is superfluous in this case.

Algebraic space-time block codes. At higher rates of transmission, more degrees of freedom must be utilized to achieve adequate performance. A reasonable approach is to employ the encoding matrices of perfect codes that are "full rate" (i.e., modulate all degrees-of-freedom), especially since it was shown in [1] that these yield a constant gap to capacity. For the case of two transmit antennas, this amounts to using a precoding matrix corresponding to the golden code.

Targeting lower transmission rates, we also consider non-full rate algebraic modulations. A straightforward such scheme is to use a punctured perfect code. Another option we examine is using Multiple-Input Double-Output (MIDO) codes as described in [10].

Random space-time block code. In [4] bounds for the WC outage behavior for space-only precoding were presented, where a random precoding matrix drawn from the CUE was considered. Such precoding is assumed in this paper to be always present as the first level of precoding, as described in Section II-C.

We extend this to encopass random space-time block modulation (i.e., we draw unitary matrices of dimensions $N_t T \times N_t T$). We note that when $\mathbf{P}_{st,c}$ is drawn from the CUE, the first-level random precoding matrix $\mathbf{P}_c$ is redundant.

Figure 2 shows the distribution of the achievable rate when using random space-time CUE precoding over a 2×2 physical channel and using $T = 2$ channel uses. As can be viewed in the figure, the "tail" of the PDF decays faster than for $T = 1$, suggesting that using random precoding over a time-extended channel can be beneficial.

Figure 3 further compares the achievable rate of the IF-SIC receiver with the ML upper bound, for the case of CUE precoding. As can be seen, the "tail" of PDF of the rate achievable with the IF-SIC receiver (i.e., the outage rate for low outage probabilities) is very close to that corresponding to the the ML bound.

IV. Simulation Results

We have simulated the performance of the candidate schemes for the cases of two and four transmit antenna systems. In implementing integer forcing, for complexity reasons, we have used the LLL algorithm for performing lattice reduction.

A. $N_r \times 2$ channels

Figure 4 compares the attainable efficiency at 1% outage probability $\eta_{0.01}(C_{WI}, \mathbf{P}_{st,c})$ for several options of space-time precoding schemes, where the receiver employs IF-SIC.

First, as a baseline for comparison, we show the efficiency of space-only CUE precoded integer forcing (with no time extension) as considered in [4]. The ML-based upper bound on efficiency (for the same precoding ensemble) is also plotted. Note that the two curves are quite close.

We also simulate several options for space-time modulation over a time-extended channel of two channel uses. Namely, we consider Alamouti modulation (transmitting one stream per channel use), CUE precoding (with $T = 2$) and modulation using the encoding matrix of the golden code. For CUE precoding, the ML bound is also depicted. As expected, at low rates Alamouti modulation performs best, achieving 100% efficiency at vanishing rates. From rates starting at just below 9 bits per channel use, random CUE precoding outperforms Alamouti when using an IF-SIC receiver. We note that had we targeted much smaller outage probabilities and higher transmission rates, golden code precoding would have outperformed random CUE precoding.

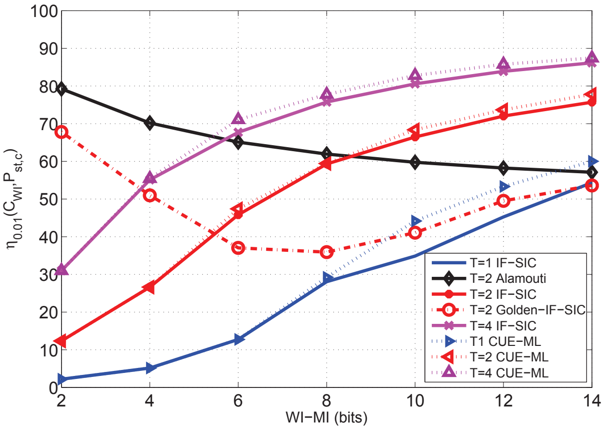

Fig. 4. Numerical results for $N_r \times 2$ channels.

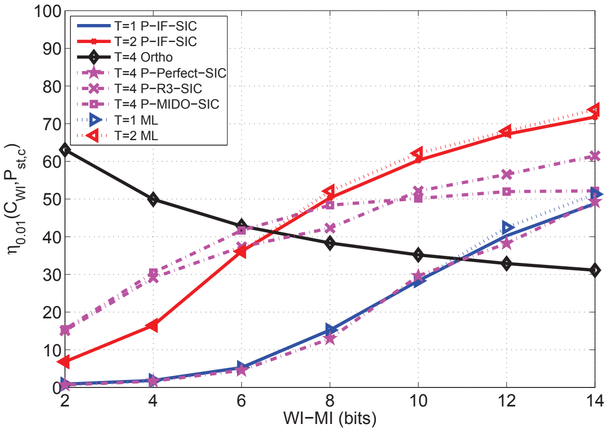

Fig. 5. Numerical results for $N_r \times 4$ channels.

We next simulated the performance of CUE precoded integer forcing over a $T = 4$ time-extended channel, further improving performance, and outperforming Alamouti modulation for rates greater than 6 bits per channel use.

B. $N_r \times 4$ channels

A similar comparison is conducted for a channel with four transmit antennas, the results appearing in Figure 5. We considered precoding schemes with "modulation rates" ranging from $3/4$ to 4 streams per channel use.

For spatial precoding only ($T = 1$) and for precoding over $T = 2$ channel uses, the performance of CUE precoding is simulated as well as the ML upper bounds. Note that in both cases, the modulation rate is 4. The results for $T = 2$ are notably superior to those for $T = 1$.

We further consider precoding over $T = 4$ channel uses. Here, we simulate OSTBC precoding (modulation rate $3/4$), perfect code modulation (full rate), modulation using a punctured perfect code (modulation rate 3) as well as via a MIDO code (modulation rate 2).

As expected, at low rates, the best performance is attained by OSTBC modulation. There is a small range of rates for which MIDO modulation is best, beyond which CUE precoding over $T = 2$ channel uses yields the best performance.

We believe that further improvement in performance is possible by applying CUE precoding over $T = 3$ channel uses. Simulation results however did not provide this gain (although the ML bound did). We believe that this due to the limitations of the LLL algorithm.

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